

Short Course on Experimental Dynamic Substructuring

Module #2: General theory (Part 1)



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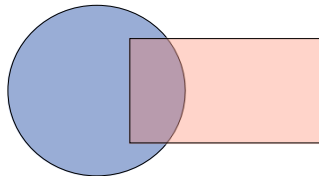
Short Course Notes For:

February 8, 2020, IMAC, Houston, TX, USA

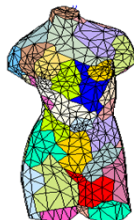
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Some history – Divide and Conquer

- The first idea of decoupling and re-assembling to solve a problem was by H. Schwarz, in 1890, who wanted to prove the existence and uniqueness of the solution to the Laplace equation in a complex domain:



- Later, with the work of Courant (1943), the idea of subdivision was used to define Ritz approximation per elements. Finite Elements were born ...



P. Gosselet – ENS Cachan

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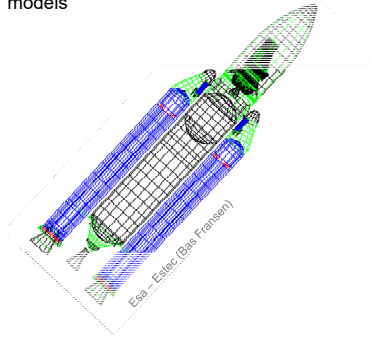


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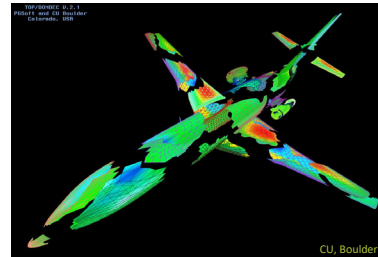
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Some history – Divide and Conquer

- In the 60's the idea of partitioning a finite element model in substructures was proposed to reduce the complexity of the models



- In the 80's, in order to speed-up the solution of linear algebraic systems, like in $Ku=F$, the computational domain was cut in sub-domains in order to share work amongst several CPUs (parallel computing). That was the start of *Domain Decomposition*.



- End of the 60's, but especially in the 80's, decomposing a structure in experimental mechanics was applied in order to simplify the testing. Today new interest thanks to new understanding of methods, better sensors and acquisition.

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Different Domains to apply substructuring [1]

Depending on how the dynamics of the components are described

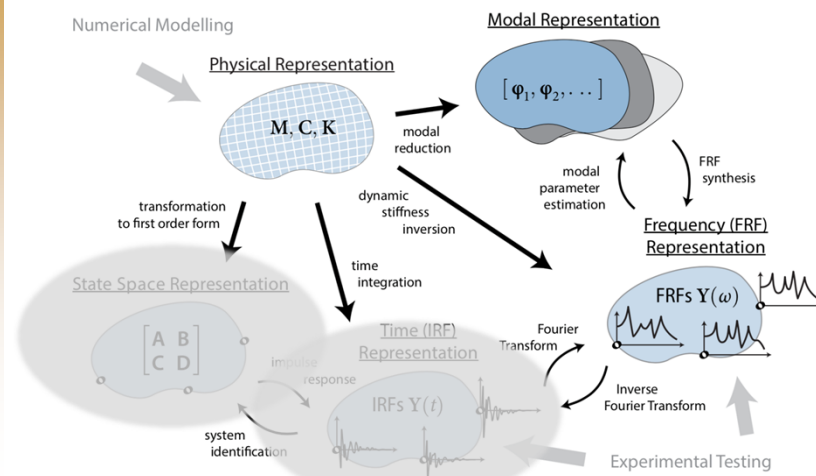


Fig. 2.1: Schematic overview of substructuring domains

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Outline

- **Coupling the physical model**
 - 3-field form
 - Primal assembly
 - Dual assembly
 - Summary & Remarks
- **Dual coupling in modal space**
- **Exercises**
- **Coupling in the frequency domain**
 - Assembled admittance
 - Example
 - Summary & Remarks
- **Weak interface compatibility**
 - By projection
 - By filtering
- **Summary**

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References and bibliography

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Coupling the physical model

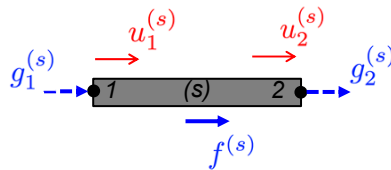
- Dynamic equation of one substructure (s)

*link forces on interface
(to be determined from global dynamics)*

$$M^{(s)} \ddot{u}^{(s)}(t) + C^{(s)} \dot{u}^{(s)}(t) + K^{(s)} u^{(s)}(t) = f^{(s)}(t) + g^{(s)}(t)$$

externally applied forces

- Example of bars.



$$K^{(s)} = k^{(s)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$M^{(s)} = m^{(s)} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$u^{(s)} = \begin{bmatrix} u_1^{(s)} \\ u_2^{(s)} \end{bmatrix}$$

$$g^{(s)} = \begin{bmatrix} g_1^{(s)} \\ g_2^{(s)} \end{bmatrix}$$

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Coupling the physical model

- Block-matrix** notation for n substructures

$$M^{(s)} \ddot{u}^{(s)}(t) + C^{(s)} \dot{u}^{(s)}(t) + K^{(s)} u^{(s)}(t) = f^{(s)}(t) + g^{(s)}(t)$$

$$M\ddot{u} + C\dot{u} + Ku = f + g$$

unassembled system

$$M \triangleq \text{diag} (M^{(1)} , \dots , M^{(n)}) = \begin{bmatrix} M^{(1)} & \cdot & \cdot \\ \cdot & \ddots & \cdot \\ \cdot & \cdot & M^{(n)} \end{bmatrix}$$

$$C \triangleq \text{diag} (C^{(1)} , \dots , C^{(n)})$$

$$K \triangleq \text{diag} (K^{(1)} , \dots , K^{(n)})$$

$$u \triangleq \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(n)} \end{bmatrix}, f \triangleq \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(n)} \end{bmatrix}, g \triangleq \begin{bmatrix} g^{(1)} \\ \vdots \\ g^{(n)} \end{bmatrix}$$

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Coupling the physical model: 3-field form

- The system is said to be **assembled** when **2 interface conditions** are met:

- Compatibility of the displacements on the interface

$$Bu = \mathbf{0}$$

← signed Boolean matrix (0,1,-1)

- Equilibrium of the interface forces (actio-reactio)

$$L^T g = \mathbf{0}$$

← Boolean matrix with (0,1)

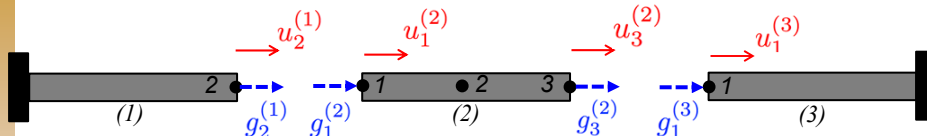
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Coupling the physical model: 3-field form

- Example of bars.



Compatibility on each interface

interface connections

$$\begin{bmatrix} \text{# unassembled dofs} \\ \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_3^{(2)} \\ u_1^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Bu = \mathbf{0}$$

Computes jump on interface

Equilibrium of internal forces on each assembled dof

assembled dof

$$\begin{bmatrix} \text{# unassembled dofs} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} g_2^{(1)} \\ g_1^{(2)} \\ g_2^{(2)} \\ g_3^{(2)} \\ g_1^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$L^T g = \mathbf{0}$$

Sums up forces on all matching dofs

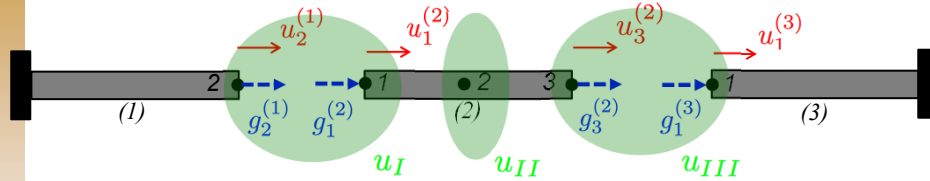
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Coupling the physical model: 3-field form

Note: L is also used to define a unique set of assembled/global dofs



$$L^T g = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} g_2^{(1)} \\ g_1^{(1)} \\ g_2^{(2)} \\ g_3^{(2)} \\ g_1^{(3)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} u_2^{(1)} \\ u_1^{(1)} \\ u_2^{(2)} \\ u_3^{(2)} \\ u_1^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_I \\ u_{II} \\ u_{III} \end{bmatrix} = L u_{glob}$$

and since global dofs are per definition compatible:

$$B u_{compatible} = B L u_{glob} = \mathbf{0} \quad \forall u_{glob} \Rightarrow \boxed{BL = \mathbf{0}} \\ L = \text{null}(B)$$

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Coupling the physical model: 3-field form

Summarizing:

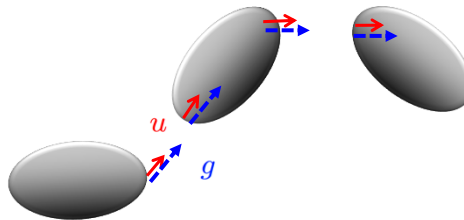
- an assembled system is described by

$$\begin{cases} M\ddot{u} + C\dot{u} + Ku = f + g \\ Bu = \mathbf{0} \\ L^T g = \mathbf{0} \end{cases} \begin{array}{l} \rightarrow \text{local equilibrium} \\ \rightarrow \text{interface compatibility} \\ \rightarrow \text{equilibrium of connecting forces} \end{array}$$

3-field formulation, using block-matrices of unassembled quantities

- Additional property

$$BL = \mathbf{0}$$



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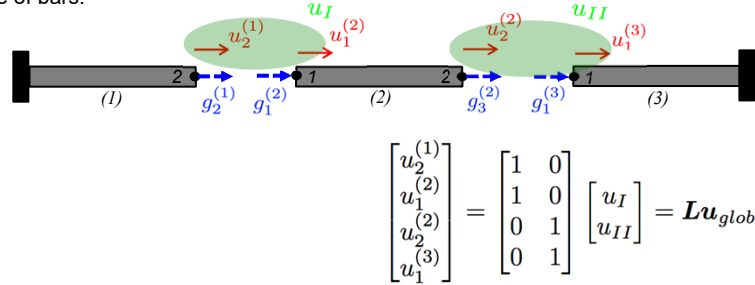
Coupling the physical model: Primal assembly

- Let us now write the assembled problem by considering from the start only a set of **dofs** that are compatible, i.e. originating from a **common/global assembled** set

$$\mathbf{u}_{glob} \quad \text{such that} \quad \mathbf{u} = \mathbf{L}\mathbf{u}_{glob}$$

In other words: the substructure $\mathbf{u}^{(s)}$ are picked from a unique global set \mathbf{u}_{glob}

- Example of bars.



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Coupling the physical model: Primal assembly

$$\mathbf{u}_{glob} \quad \text{such that} \quad \mathbf{u} = \mathbf{L}\mathbf{u}_{glob}$$

$$\begin{cases} \mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} + \mathbf{g} \\ \mathbf{B}\mathbf{u} = \mathbf{0} \text{ naturally satisfied} \\ \mathbf{L}^T \mathbf{g} = \mathbf{0} \end{cases} \rightarrow \begin{cases} \mathbf{M}\mathbf{L}\ddot{\mathbf{u}}_{glob} + \mathbf{C}\mathbf{L}\dot{\mathbf{u}}_{glob} + \mathbf{K}\mathbf{L}\mathbf{u}_{glob} = \mathbf{f} + \mathbf{g} \\ \mathbf{L}^T \mathbf{g} = \mathbf{0} \end{cases}$$

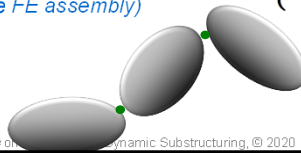
and one can eliminate the unknown interface forces by assembling the local equilibriums and the interface:

$$\mathbf{L}^T (\mathbf{M}\mathbf{L}\ddot{\mathbf{u}}_{glob} + \mathbf{C}\mathbf{L}\dot{\mathbf{u}}_{glob} + \mathbf{K}\mathbf{L}\mathbf{u}_{glob} = \mathbf{f} + \mathbf{g})$$

$$\tilde{\mathbf{M}}\ddot{\mathbf{u}}_{glob} + \tilde{\mathbf{C}}\dot{\mathbf{u}}_{glob} + \tilde{\mathbf{K}}\mathbf{u}_{glob} = \tilde{\mathbf{f}}$$

Primally assembled (like FE assembly)

$$\begin{cases} \tilde{\mathbf{M}} \triangleq \mathbf{L}^T \mathbf{M} \mathbf{L} \\ \tilde{\mathbf{C}} \triangleq \mathbf{L}^T \mathbf{C} \mathbf{L} \\ \tilde{\mathbf{K}} \triangleq \mathbf{L}^T \mathbf{K} \mathbf{L} \\ \tilde{\mathbf{f}} \triangleq \mathbf{L}^T \mathbf{f} \end{cases}$$



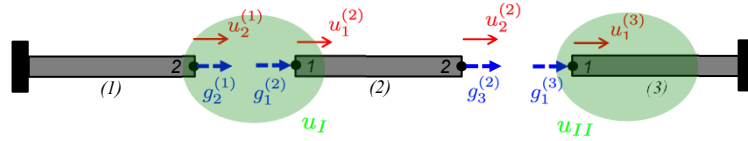
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Coupling the physical model: Primal assembly

- Example of bars.



$$\bar{K} \triangleq L^T K L = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k^{(1)} & 0 & 0 & 0 \\ 0 & k^{(2)} & -k^{(2)} & 0 \\ 0 & -k^{(2)} & k^{(2)} & 0 \\ 0 & 0 & 0 & k^{(3)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} k^{(1)} + k^{(2)} & -k^{(2)} \\ -k^{(2)} & k^{(2)} + k^{(3)} \end{bmatrix}$$

exactly like in Finite Elements.

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Coupling the physical model: Dual assembly

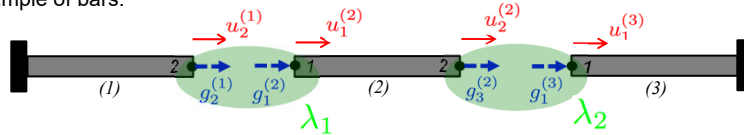
- Starting again from the 3-field formulation, let us now write the assembled problem by considering from the start only a set of **interface forces** that are in equilibrium, i.e. **equal an opposite**:

$$\begin{cases} M\ddot{u} + C\dot{u} + Ku = f + g \\ Bu = 0 \\ L^T g = 0 \end{cases}$$

$$\lambda \text{ such that } g = -B^T \lambda$$

In other words: the interface forces are $\pm \lambda$ on each side of the interface

- Example of bars.



$$g = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = -B^T \lambda$$

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Coupling the physical model: Dual assembly

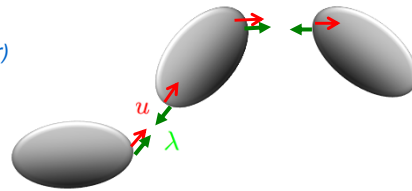
$$\lambda \text{ such that } g = -B^T \lambda$$

$$\begin{cases} M\ddot{u} + C\dot{u} + Ku = f + g \\ Bu = 0 \\ \mathcal{L}^T g = 0 \text{ naturally satisfied} \end{cases} \rightarrow \begin{cases} M\ddot{u} + C\dot{u} + Ku + B^T \lambda = f \\ Bu = 0 \end{cases}$$

and in block matrix form

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \lambda \end{bmatrix} + \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \lambda \end{bmatrix} + \begin{bmatrix} K & B \\ B^T & 0 \end{bmatrix}^T \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

Dually assembled
(like in constrained multibody formulation,
 λ = Lagrange multiplier)



Note:

The compatibility condition could also be written on the velocities or accelerations. Careful for drift.

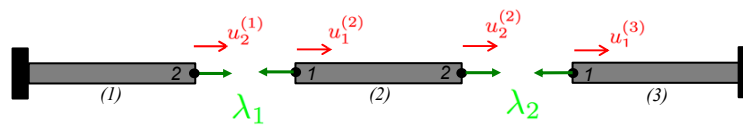
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Coupling the physical model: Dual assembly

- Example of bars.



$$\begin{bmatrix} K & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} k^{(1)} & 0 & 0 & 0 \\ 0 & k^{(2)} & -k^{(2)} & 0 \\ 0 & -k^{(2)} & k^{(2)} & 0 \\ 0 & 0 & 0 & k^{(3)} \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ \lambda_1 \\ \lambda_2 \end{bmatrix}$$

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Coupling the physical model: SUMMARY

$$\begin{cases} M\ddot{u} + C\dot{u} + Ku = f + g \\ Bu = 0 \\ L^T g = 0 \end{cases}$$

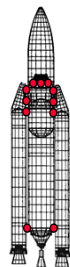
3-field

All mathematically equivalent !

$$\tilde{M}\ddot{u}_{glob} + \tilde{C}\dot{u}_{glob} + \tilde{K}u_{glob} = \tilde{f}$$

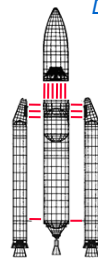
$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{u} \\ \lambda \end{bmatrix} + \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \lambda \end{bmatrix} + \begin{bmatrix} K & B \\ B^T & 0 \end{bmatrix}^T \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

Primal



$$\begin{bmatrix} K^{(1)} & 0 \\ 0 & K^{(N)} \end{bmatrix} \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(N)} \end{bmatrix} = \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(N)} \end{bmatrix}$$

Dual



$$\begin{bmatrix} K^{(1)} & 0 & B^{(1)T} \\ & \ddots & \\ 0 & K^{(N)} & B^{(N)T} \\ B^{(1)} & \dots & B^{(N)} & 0 \end{bmatrix} \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(N)} \\ \lambda \end{bmatrix} = \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(N)} \\ 0 \end{bmatrix}$$

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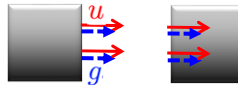
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Coupling the physical model: REMARKS

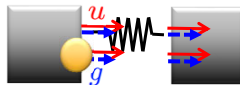
- When enforcing interface equilibrium and compatibility, we have assumed that the interface has no local dynamics:

$$\begin{aligned} Bu &= 0 \\ L^T g &= 0 \end{aligned}$$



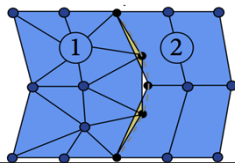
If "simple" interface flexibility, damping or mass is present at the interface, then the interface conditions must be slightly altered

$$\begin{aligned} Bu &= \cancel{0} \\ L^T g &= \cancel{0} \end{aligned}$$



and modified primal and dual assembled systems can be derived [2,3]

- If incompatible interface, the dual assembly can still be applied (B non Boolean) e.g. [5]



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Dual coupling in modal space

- How do the assembled equations look if

each substructure is approximated by a
reduced set of (modal) shapes stored in the columns of $R^{(s)}$

$$u^{(s)} \simeq R^{(s)} \eta^{(s)} \quad \text{in local equilibria}$$

$$M^{(s)} \ddot{u}^{(s)}(t) + C \dot{u}^{(s)}(t) + K^{(s)} u^{(s)}(t) = f^{(s)}(t) + g^{(s)}(t)$$

$$M^{(s)} R^{(s)} \ddot{\eta}^{(s)} + C^{(s)} R^{(s)} \dot{\eta}^{(s)} + K^{(s)} R^{(s)} \eta^{(s)} = f^{(s)} + g^{(s)} + r^{(s)}$$

or in block notations

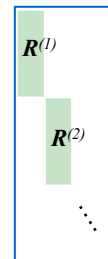
Equilibrium error due to response approximation

$$u \simeq R \eta$$

$$R \triangleq \text{diag} (R^{(1)} , \dots , R^{(n)})$$

$$M \ddot{u} + C \dot{u} + K u = f + g$$

$$M R \ddot{\eta} + C R \dot{\eta} + K R \eta = f + g + r$$



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Dual coupling in modal space

$$\mathbf{u} \simeq \mathbf{R}\boldsymbol{\eta} \quad \mathbf{M}\mathbf{R}\ddot{\boldsymbol{\eta}} + \mathbf{C}\mathbf{R}\dot{\boldsymbol{\eta}} + \mathbf{K}\mathbf{R}\boldsymbol{\eta} = \mathbf{f} + \mathbf{g} + \mathbf{r}$$

Equilibrium error due to response approximation

Asking that the residual force creates no work in the approximation space (as done in reduction techniques – virtual work),

$$\mathbf{R}^T (\mathbf{M}\mathbf{R}\ddot{\boldsymbol{\eta}} + \mathbf{C}\mathbf{R}\dot{\boldsymbol{\eta}} + \mathbf{K}\mathbf{R}\boldsymbol{\eta} - \mathbf{f} - \mathbf{g}) = \mathbf{0}$$

$$\rightarrow \mathbf{M}_m \ddot{\boldsymbol{\eta}} + \mathbf{C}_m \dot{\boldsymbol{\eta}} + \mathbf{K}_m \boldsymbol{\eta} = \mathbf{f}_m + \mathbf{g}_m$$

where

$$\begin{cases} \mathbf{M}_m \triangleq \mathbf{R}^T \mathbf{M} \mathbf{R} \\ \mathbf{C}_m \triangleq \mathbf{R}^T \mathbf{C} \mathbf{R} \\ \mathbf{K}_m \triangleq \mathbf{R}^T \mathbf{K} \mathbf{R} \\ \mathbf{f}_m \triangleq \mathbf{R}^T \mathbf{f} \\ \mathbf{g}_m \triangleq \mathbf{R}^T \mathbf{g} \end{cases} \quad \text{are the matrices in the representation space (reduced)}$$

The compatibility condition $\mathbf{B}\mathbf{u} = \mathbf{0}$ transforms into $\mathbf{B}_m \boldsymbol{\eta} = \mathbf{0}$

where $\mathbf{B}_m \triangleq \mathbf{B} \mathbf{R}$

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Dual coupling in modal space

Following the same reasoning as for the general case, one can write the dually assembled problem in the representation space [1]

$$\begin{bmatrix} \mathbf{M}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{\eta}} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\eta}} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_m & \mathbf{B}_m \\ \mathbf{B}_m^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_m \\ \mathbf{0} \end{bmatrix}$$

REMARKS:

- Looks very similar to the Dual problem in the physical space but ...
 - ... the matrices are reduced and expressing the physical properties in the representation space
 - ... the unknowns are now the amplitudes of the representation vectors
- It is important to build the reduction space \mathbf{R} on shapes that can properly represent the response of the assembled system (see beam example by Matt Allen)
 - some methods use modes of the system when assembled with a dummy mimicking roughly neighboring structures (see the transmission simulator idea)

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Dual coupling in representation space (Modal)

$$\begin{bmatrix} M_m & \mathbf{o} \\ \mathbf{o} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \ddot{\eta} \\ \lambda \end{bmatrix} + \begin{bmatrix} C_m & \mathbf{o} \\ \mathbf{o} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \dot{\eta} \\ \lambda \end{bmatrix} + \begin{bmatrix} K_m & B_m \\ B_m^T & \mathbf{o} \end{bmatrix}^T \begin{bmatrix} \eta \\ \lambda \end{bmatrix} = \begin{bmatrix} f_m \\ \mathbf{o} \end{bmatrix}$$

REMARKS (continued):

- If the approximation space is the modal space of the substructures (free interface),

$$\mathbf{u}^{(s)} = \Phi^{(s)} \eta^{(s)} \quad \text{where} \quad K^{(s)} \Phi^{(s)} = M^{(s)} \Phi^{(s)} \Omega^{(s)^2}$$

then in frequency domain (with proportional damping)

$$\begin{bmatrix} \begin{bmatrix} -\omega^2 \mathbf{I} + i\omega 2\zeta^{(1)} \Omega^{(1)} + \Omega^{(1)^2} & 0 \\ 0 & \ddots \end{bmatrix} & \begin{bmatrix} \Phi^{(1)T} B^{(1)T} \\ \vdots \end{bmatrix} \\ \begin{bmatrix} B^{(1)} \Phi^{(1)} & \dots \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} \eta^{(1)} \\ \vdots \\ \lambda \end{bmatrix} = \begin{bmatrix} \Phi^{(1)T} f^{(1)} \\ \vdots \\ 0 \end{bmatrix}$$

- One major issue: the condition $BR\eta = B_m\eta = \mathbf{o}$ requires full compatibility on the interface. Since each substructure is represented by a small number of local modes, this might lead to **locking** (the compatibility can be satisfied only when 0 response)
→ need for weakening the interface compatibility

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Coupling in Frequency Domain

- Exactly the same assembly procedures

$$\mathbf{u}^{(s)}(t) = \int_{-\infty}^{\infty} \hat{\mathbf{u}}^{(s)}(\omega) e^{-i\omega t} d\omega$$

$$\begin{cases} M\ddot{\mathbf{u}} + C\dot{\mathbf{u}} + K\mathbf{u} = \mathbf{f} + \mathbf{g} \\ B\mathbf{u} = \mathbf{0} \\ L^T \mathbf{g} = \mathbf{0} \end{cases}$$

$$\begin{cases} Z(\omega)\mathbf{u}(j\omega) = \mathbf{f}(\omega) + \mathbf{g}(\omega) \\ B\mathbf{u}(\omega) = \mathbf{0} \\ L^T \mathbf{g}(\omega) = \mathbf{0} \end{cases} \quad \text{3-field}$$

$$Z(j\omega) \triangleq -\omega^2 M + j\omega C + K$$

Structural impedance matrix

Primal

$$\tilde{Z}q = \tilde{f}$$

With the assembled impedance

$$\tilde{Z} \triangleq L^T Z L$$

Dual

$$\begin{bmatrix} Z & B^T \\ B & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}$$

where Z is the block-matrix of impedances of each individual substructure (i.e. free interface)

Commonly called
Frequency Based Substructuring or FBS

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Coupling in Frequency Domain: Assembled Admittance

The dual assembly is usually preferred when the substructure data are coming from measurements. Why ?

$$\begin{bmatrix} Z & B^T \\ B & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}$$

$$\rightarrow \mathbf{u} = Y(\mathbf{f} - B^T \lambda)$$

$$\text{where } Y(j\omega) \triangleq Z^{-1} = (-\omega^2 M + j\omega C + K)^{-1}$$

Block-matrix of substructure admittances

$$B(Y\mathbf{f} - YB^T \lambda) = \mathbf{0}$$

$$(BYB^T)\lambda = BY\mathbf{f}$$

Dual interface problem:
"how much interface force is needed to close the interface gap resulting from external forces"

$$\begin{aligned} \mathbf{u} &= Y\mathbf{f} - YB^T(BYB^T)^{-1}BY\mathbf{f} \\ &= (Y - YB^T(BYB^T)^{-1}BY)\mathbf{f} \end{aligned}$$

=Assembled Admittance

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Coupling in Frequency Domain: Assembled Admittance

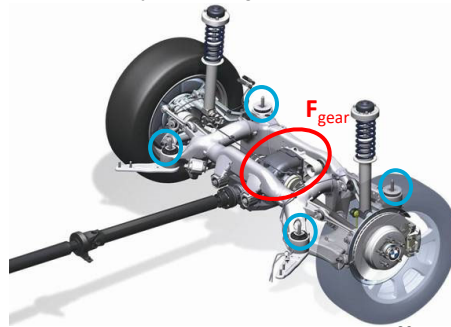
Summarizing:

If one knows the admittance of each substructure free on its interface, then the dynamic response of the assembled system is

$$u = (Y - YB^T(BYB^T)^{-1}BY)f$$

i.e. one can compute the admittance of the assembled system using local admittances !

→ useful in experimental dynamics
where admittances of non-connected
components are measured.



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Coupling in Frequency Domain: Example

- Example of bars.



$$u = (Y - YB^T(BYB^T)^{-1}BY)f$$

$$Y = \begin{bmatrix} Y_{22}^{(1)} & Y_{23}^{(1)} & 0 & 0 \\ Y_{32}^{(1)} & Y_{33}^{(1)} & 0 & 0 \\ 0 & 0 & Y_{11}^{(2)} & Y_{12}^{(2)} \\ 0 & 0 & Y_{21}^{(2)} & Y_{22}^{(2)} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$$

$$u = \begin{bmatrix} u_2^{(1)} \\ u_3^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \end{bmatrix}$$

$$\begin{bmatrix} u_2^{(1)} \\ u_3^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \end{bmatrix} = \left\{ \begin{bmatrix} Y_{22}^{(1)} & Y_{23}^{(1)} & 0 & 0 \\ Y_{32}^{(1)} & Y_{33}^{(1)} & 0 & 0 \\ 0 & 0 & Y_{11}^{(2)} & Y_{12}^{(2)} \\ 0 & 0 & Y_{21}^{(2)} & Y_{22}^{(2)} \end{bmatrix} - \begin{bmatrix} Y_{23}^{(1)} \\ Y_{33}^{(1)} \\ -Y_{11}^{(2)} \\ -Y_{21}^{(2)} \end{bmatrix} \left(Y_{33}^{(1)} + Y_{11}^{(2)} \right)^{-1} \begin{bmatrix} Y_{32}^{(1)} & Y_{33}^{(1)} & -Y_{11}^{(2)} & -Y_{12}^{(2)} \end{bmatrix} \right\} \begin{bmatrix} f_2^{(1)} \\ f_3^{(1)} \\ f_1^{(2)} \\ f_2^{(2)} \end{bmatrix}$$

assembled interface flexibility

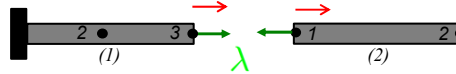
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Coupling in Frequency Domain: Example

- Example of bars.



$$u = (Y - YB^T(BYB^T)^{-1}BY)f$$

$$Y = \begin{bmatrix} Y_{22}^{(1)} & Y_{23}^{(1)} & 0 & 0 \\ Y_{32}^{(1)} & Y_{33}^{(1)} & 0 & 0 \\ 0 & 0 & Y_{11}^{(2)} & Y_{12}^{(2)} \\ 0 & 0 & Y_{21}^{(2)} & Y_{22}^{(2)} \end{bmatrix}$$

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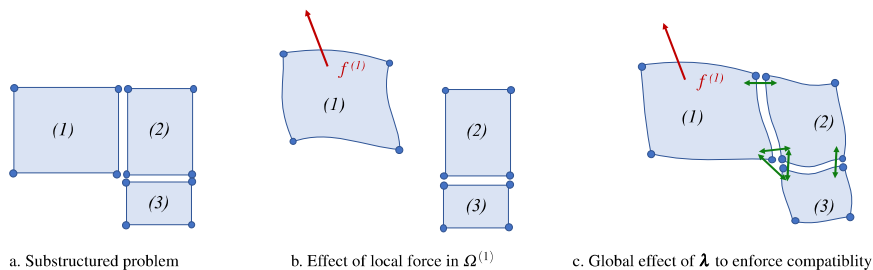
response to applied forces
response to interface forces
incompatibility between substructures when not connected
interface force λ needed for compatibility

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Coupling in Frequency Domain: Interpretation of assembled admittance



a. Substructured problem

b. Effect of local force in $\Omega^{(1)}$

c. Global effect of λ to enforce compatibility

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Coupling in Frequency Domain: Summary

Summary to remember !

$$u = (Y - YB^T(BYB^T)^{-1}BY)f$$

where Y contains in its blocks the uncoupled admittances

where B is signed Boolean and describes the connection topology

$$u = (\underbrace{Y}_{\text{response to applied forces}} - \underbrace{YB^T}_{\text{interface force } \lambda} (\underbrace{BYB^T}_{\text{incompatibility between substructures when no connection}})^{-1} \underbrace{BY}_{\text{response to interface forces}})f$$

incompatibility between substructures
when no connection

interface force λ needed for compatibility

response
to applied forces

response
to interface forces

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Coupling in Frequency Domain: Remarks

$$u = (Y - YB^T(BYB^T)^{-1}BY)f$$

■+ Systematic assembly for any number of substructures and any topology

■- needs admittances for all interface dofs ...

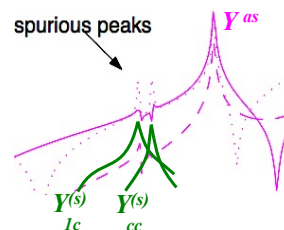
- - not always possible to place sensors on interface
- - not always possible to measure **all** dofs on interface

■- implies *inverting* the assembled interface flexibility

→ explosion of measurement errors [6]

e.g. measurement noise

e.g. poles in $Y^{(s)}$ not identical for all coefficient [7]



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Coupling in Frequency Domain: Remarks

$$u = (Y - YB^T(BYB^T)^{-1}BY)f$$

- + Systematic assembly for any number of substructures and any topology
 - needs admittances for all interface dofs ...
 - - not always possible to place sensors on interface
 - - not always possible to measure **all** dofs on interface
 - implies *inverting* the assembled interface flexibility
 - explosion of measurement errors [6]
 - e.g. measurement noise
 - e.g. poles in $Y^{(s)}$ not identical for all coefficient [7]
- expansion of measurements in the vicinity of interface using local shapes (e.g. Serep).
- SVD filtering of data, e.g. [8,13]
- Modal synthesis of data (if low modal density)
- Correction with FE model [9]
- Cleaning Y as *a posteriori* [10]
- Weakening of interface compatibility → next sections

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Weak interface compatibility

Enforcing the strong compatibility for the entire interface can lead to serious problems when (see earlier discussions)

- the admittance contains measurement errors and the interface is stiff
- when the substructures are approximated by a limited number of representative shapes that do not necessarily match on the interface (locking)

➡ 2 different techniques to relax the interface compatibility (**weakening**)

1. requiring that only a projection of the interface gap is 0
→ **Projected compatibility**
2. projecting the interface displacements on each side of the interface on a common reduced space and enforcing compatibility only for that space
→ **Filtering**

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Weak interface compatibility: Projected Compatibility

Starting from the frequency representation with physical dofs (would be similar for modal representation):

$$\begin{bmatrix} \mathbf{Z} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{o} \end{bmatrix}$$

In order to weaken the compatibility condition, we will require only that

the projection of the interface gap on a small number of interface modes is 0

$$\mathbf{\Gamma}_I^T (\mathbf{B}\mathbf{u}) = 0$$

where $\mathbf{\Gamma}_I$ contains the interface projection shapes.

For instance if $\mathbf{\Gamma}_I = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \dots$



..... one imposes only an average compatibility

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Weak interface compatibility: Projected Compatibility

One can show that the dual problem then writes

$$\begin{bmatrix} \mathbf{Z} & \mathbf{B}^T \mathbf{\Gamma}_I \\ \mathbf{\Gamma}_I^T \mathbf{B} & \mathbf{o} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \lambda_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{o} \end{bmatrix}$$

- small number of compatibility conditions on the interface (projected)
- small number of interface force variables λ_Γ
- can be interpreted as if the compatibility is enforced through interface forces that are built on the interface projection shapes: $\lambda \simeq \mathbf{\Gamma}_I \lambda_\Gamma$

- Similarly, if the substructure displacements are approximated in a representation space:

$$\mathbf{u} \simeq \mathbf{R}\boldsymbol{\eta} \quad \text{and} \quad \begin{bmatrix} \mathbf{Z}_m & \mathbf{B}_m^T \mathbf{\Gamma}_I \\ \mathbf{\Gamma}_I^T \mathbf{B}_m & \mathbf{o} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \lambda_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{f}_m \\ \mathbf{o} \end{bmatrix}$$

in which the displacements **and** the interface forces are approximated.

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Weak interface compatibility: Filtering

On the second approach, one fits the interface displacement on both sides of the interface on a common displacement shape.

Calling Φ_Γ the assumed modes for the interface displacements,

$$\mathbf{u}^{(s)} = \begin{bmatrix} \mathbf{u}_i^{(s)} \\ \mathbf{u}_c^{(s)} \end{bmatrix} = \begin{bmatrix} \mathbf{u}_i^{(s)} \\ \Phi_\Gamma \alpha^{(s)} + \tilde{\mathbf{u}}_c^{(s)} \end{bmatrix}$$

→ internal dofs
→ connecting dofs
→ common interface modes → part not representable in Φ_Γ

One will enforce a weak compatibility by asking only that

the amplitudes of Φ_Γ should be equal on each side of the interface: $B\alpha = 0$

$$\mathbf{u}_c^{(s)} \simeq \Phi_\Gamma \alpha^{(s)}$$

and by least square

$$\alpha^{(s)} = (\Phi_\Gamma^T \Phi_\Gamma)^{-1} \Phi_\Gamma^T \mathbf{u}_c^{(s)}$$

$$\Rightarrow B(\Phi_\Gamma^T \Phi_\Gamma)^{-1} \Phi_\Gamma^T \mathbf{u}_c = B\mathbf{F}\mathbf{u} = 0$$

↙ filter that removes everything that is not in the space of Φ_Γ

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Weak interface compatibility: Filtering

Finally the dual problem writes

$$\begin{bmatrix} \mathbf{Z} & \mathbf{F}^T \mathbf{B}^T \\ \mathbf{BF} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix}$$

- similarly if the displacements are approximated in a modal space

$$\begin{bmatrix} \mathbf{Z}_m & \mathbf{R}^T \mathbf{F}^T \mathbf{B}^T \\ \mathbf{BFR} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} \mathbf{f}_m \\ \mathbf{0} \end{bmatrix}$$

- λ_α has the meaning of generalized forces associated to the interface displacements
- An important application: when Φ_Γ contains the rigid body motion of the interface, then λ_α are the 6 forces and moments resultants at the interface
 - often used for rigid interfaces (bolts, brackets ...), allowing to construct rotational dofs
 - in that case the method is also called EMPC (Explicit Multi-Point connection) or virtual point transformation [11,12]

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Summary

$$\begin{bmatrix} Z & B^T \\ B & \mathbf{o} \end{bmatrix} \begin{bmatrix} u \\ \lambda \end{bmatrix} = \begin{bmatrix} f \\ \mathbf{o} \end{bmatrix} \Leftrightarrow u = \overbrace{(Y - YB^T(BYB^T)^{-1}BY)}^{\text{assembled admittance}} f$$

or

$$\begin{bmatrix} Z_m & B_m^T \\ B_m & \mathbf{o} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda \end{bmatrix} = \begin{bmatrix} f_m \\ \mathbf{o} \end{bmatrix} \quad \text{in reduced representation space } u \simeq R\eta$$

$$Z_m = R^T Z R$$

Weakening of compatibility by *projection*:

$$\lambda \simeq \Gamma_I \lambda_\Gamma$$

$$\begin{bmatrix} Z & B^T \Gamma_I \\ \Gamma_I^T B & \mathbf{o} \end{bmatrix} \begin{bmatrix} u \\ \lambda_\Gamma \end{bmatrix} = \begin{bmatrix} f \\ \mathbf{o} \end{bmatrix}$$

or

$$\begin{bmatrix} Z_m & B_m^T \Gamma_I \\ \Gamma_I^T B_m & \mathbf{o} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda_\Gamma \end{bmatrix} = \begin{bmatrix} f_m \\ \mathbf{o} \end{bmatrix}$$

Weakening of compatibility by *filtering*:

$$u_c^{(s)} \simeq \Phi_\Gamma \alpha^{(s)}$$

$$F = (\Phi_\Gamma^T \Phi_\Gamma)^{-1} \Phi_\Gamma^T$$

$$\begin{bmatrix} Z & F^T B^T \\ BF & \mathbf{o} \end{bmatrix} \begin{bmatrix} u \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} f \\ \mathbf{o} \end{bmatrix}$$

or

$$\begin{bmatrix} Z_m & R^T F^T B^T \\ BFR & \mathbf{o} \end{bmatrix} \begin{bmatrix} \eta \\ \lambda_\alpha \end{bmatrix} = \begin{bmatrix} f_m \\ \mathbf{o} \end{bmatrix}$$

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